Euler's Equation:

A function V is said to be **homogeneou**s of **degree** n if it satisfies the Euler's Equation:

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=-nV$$

1) It can be proved that if V satisfies Euler's Equation of degree n, then the spatial derivative of V is also *homogeneous* and is of degree *n+1*. For example, the first vertical derivative of V satisfies:

$$x\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}V\right)+y\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}V\right)+z\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}V\right)=-(n+1)\left(\frac{\partial}{\partial z}V\right)$$

2) $V=\frac{1}{r}$ satisfies Euler's equation and has degree 1. It follows that the gravitational potential of a point mass $V=\frac{Gm}{r}$ also satisfies Euler's Equation with n=1. The gravitational acceleration or the magnetic potential for a dipole involves the first spatial derivative of $V=\frac{1}{r}$, thus they satisfy Euler's equation with degree 2. Similarly, potential field of other geometries involves higher derivatives of $V=\frac{1}{r}$, thus they satisfy Euler's equation with a higher order n.

Derive Equation for Euler Deconvolution:

Consider a simple potential source with center at (x_0, y_0, z_0). For this source, some gravity or total magnetic field measurements $T(x_i,y_i,z_i)$ are made at locations (x_i,y_i,z_i), with i =1,2,3,…m. From above, we see that the measurements T must satisfies Euler's equation of degree n. Thus:

$$(x_i-x_0)\frac{\partial T(x_i,y_i,z_i)}{\partial x}+(y_i-y_0)\frac{\partial T(x_i,y_i,z_i)}{\partial y}+(z_i-z_0)\frac{\partial T(x_i,y_i,z_i)}{\partial z}=-nT(x_i,y_i,z_i)$$

for all i =1,2,3,…m. If B is a constant background field, then $\Delta T_i=T(x_i,y_i,z_i)-B$ also satisfy Euler's equation of degree n. (since B is constant, $\frac{\partial \Delta T_i}{\partial x}=\frac{\partial T_i}{\partial x}$, etc.) Thus,

$$(x_i-x_0)\frac{\partial \Delta T_i}{\partial x}+(y_i-y_0)\frac{\partial \Delta T_i}{\partial y}+(z_i-z_0)\frac{\partial \Delta T_i}{\partial z}=-n\Delta T_i$$

for all i =1,2,3,…m. In matrix form, these m equations can be written as:

$$\begin{bmatrix}\frac{\partial}{\partial x}\Delta T_1 & \frac{\partial}{\partial y}\Delta T_1 & \frac{\partial}{\partial z}\Delta T_1\\ \frac{\partial}{\partial x}\Delta T_2 & \frac{\partial}{\partial y}\Delta T_2 & \frac{\partial}{\partial z}\Delta T_2\\ \vdots & \vdots & \vdots\end{bmatrix}\begin{bmatrix}x_i-x_0\\ y_i-y_0\\ z_i-z_0\end{bmatrix}=-n\begin{bmatrix}\Delta T_1\\ \Delta T_2\\ \vdots\end{bmatrix}$$

Now, the ΔT_i (i =1,2,3,…m) are measured and their derivatives can be computed, thus the first matrix on the left side and the vector on the right are known. The location of the m measurements (x_i,y_i,z_i) are also known. For a given n (structural index), the location of the source (x_0, y_0, z_0) can be solved using the Least Square Method. The m measurements form the window.
This method works well for spheres and cylinders. For other shapes, n may not be a constant because V is no longer a derivative of 1/r, but involves integration over the entire source.
***Note that the minimum depth z_0 calculated ~ grid size and Max depth ~ twice window size.

Euler deconvolution is used for rapid interpretation of **magnetic** and **gravity** data. It is particularly good at delineating contacts and rapid depth estimation. An advantage of this method in the interpretation of magnetic data is that it is insensitive to magnetic inclination, declination and remanence.

Euler's Equation for Decon Potential Field T (gravity or magnetic field) measured at (x_i, y_i, z_i):

$$(x_i - x_0)\frac{\partial T_i}{\partial x} + (y_i - y_0)\frac{\partial T_i}{\partial y} + (z_i - z_0)\frac{\partial T_i}{\partial z} = -n(T_i - B)$$

where (x_0, y_0, z_0) is the position of the source, B=value of the regional field (usually a constant), n is the structural index (SI).

Input:

1) gridded data at different location (x,y,z) e.g $T(x_1,y_1,z_1)$, $T(x_2,y_2,z_2)$, $T(x_3,y_3,z_3)$, … etc.
2) the 1st vertical derivatives of T : ($\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$, $\frac{\partial T}{\partial z}$) at all the above locations
3) SI , window size and error tolerance (used to select solution, see below).

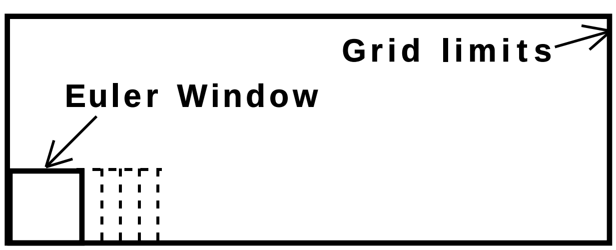
Output: location of the source (x_0, y_0, z_0) and B.

The quality of the **depth estimation** mostly depends on the choice of the proper structural index.

The **structural index** (SI) is a measure of the rate of change (fall-off rate) with distance of a field ($V = 1/r^n$). It is a function of the geometry of the causative bodies.

SI	**Magnetic Field**		**Gravity Field**
	Geologic	Cultural Model	
0.0	contact	edge of large tank	sill / dyke / step
0.5	thick step		ribbon
1.0	sill / dyke	Vertical / flat sheet	pipe
2.0	pipe	vertical drum / pipeline	sphere
3.0	sphere	tank	

Note that a 0 index implies that the field is a constant regardless of distance from the source model. This is physically impossible because the source has to approach 'infinite dimensions'. In practice, an index of 0.5 can often be used to obtain reasonable results when an index of 0 would otherwise be indicated. For a 'contact' model with gravity data, one should work with the 1st vertical derivative of the gravity data instead of the original gravity data.



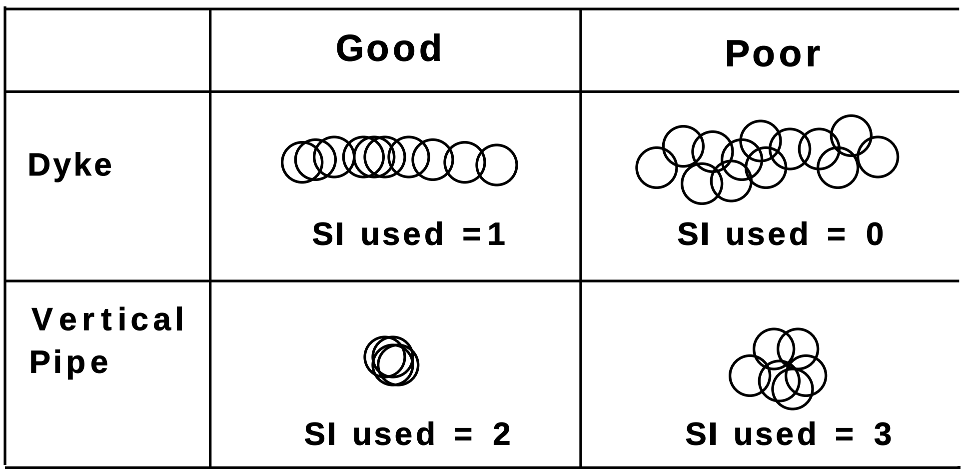
>>> window shifts one grid cell for each solution

The **window size** should be chosen with the following criteria in mind:
(i) It should be large enough to incorporate substantial variation of the field and field gradient.
(ii) It should be small enough NOT to include significant effects from multiple sources.

If the anomalies arising from different sources are so close together that both occupy any given window, poor fit statistics cause the solution to be rejected. Thus, one should keep the window as small as possible. On the other hand, broad anomalies arising from deep sources are poorly represented in a small window, and unreliable estimates of depth and position are likely.

The program uses a least squares method to solve Euler's equation simultaneously for each grid position within a window. For a 10x10 window there are 100 equations from which **the 4 unknowns (location x_0, y_0, z_0 and the background value B) and their uncertainties (standard deviations) are obtained for a specified structural index**.

A solution is recorded if the depth uncertainty of the calculated depth is less than a specified **tolerance** (typically 15% of the depth) and the solution is within a limiting distance of the center of the data window. When the solutions are plotted as circles on a map (center of circle is plotted at (x_0, y_0) and the color is determined by the depth z_0). **For a given SI and window size, a clustering of circles will be produced over anomalies - tight clustering (see below for magnetic field solutions) represent good estimates.**



If the index used is too low (e.g. if SI=0 is used for a dyke in magnetics), the depth estimate will be too shallow. If the index used is too high, the depth estimate will be too deep. Even if the index is correct, depth estimates are usually more precise for high-index sources than for low. Experience indicates that structural index within 0.25 to 0.5 of the correct value for the causative body generally yield well-focused solutions.

Acceptable results can be obtained for real data if the following selection criteria are used:

SI	Depth Uncertainty	Horizontal Uncertainty
0	20%	40%
0.5	15%	30%
1.0	10%	20%
2.0	5%	10%
3.0	5%	10%

When interpreting the results, one should ask:
i) For a given feature, is the chosen structural index correct?
ii) Are the objects of interest represented in the results?
Arriving at acceptable solutions for features of interest may involve some trial and error by adjusting the structural index, the window size and the uncertainty tolerance.